J01E.2—Betatron

Problem

A betatron is a device in which ultrarelativistic electrons are held in a circle of fixed radius R (taken to be centered on the origin in the x-y plane) by a magnetic field $B_z(r,t)$ while their energy is increased via a changing magnetic flux $d\Phi/dt = \pi R^2\, dB_{z,ave}/dt$ through the circle. Motion of the electrons perpendicular to the circle is prevented by means that need not be considered here.

Deduce the relation between the magnetic field B_z at radius R and the magnetic field $B_{z,ave}$ averaged over the area of the circle. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of $B_z, dB_{z,ave}/dt$ and R.

Hints: The electrons in this problem are ultrarelativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2 \gg 1$, where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.