M02Q.1—Driven Harmonic Oscillator

Problem

Consider the driven harmonic oscillator:

$$H(t) = \frac{1}{2}(p^2 + x^2) - \sqrt{2}f(t)x,$$

where $f(t)$ is a c-number function of time. Note that we have set the natural frequency of the oscillator ω_0, to 1. In the following you should also use $\hbar = 1$.

Defining the time evolution operator in the Schrödinger picture by,

$$|\psi(t)\rangle_S = U(t)|\psi(0)\rangle$$

we can transform to the Heisenberg picture

$$|\psi\rangle_H = U^\dagger(t)|\psi(t)\rangle_S \equiv |\psi(0)\rangle$$

and

$$O_H(t) = U^\dagger(t)O_S U(t).$$

Working in the Heisenberg picture,

a) Write down the equations of motion for the operators x and p.

b) Solve the operator equations of motion derived in a) for the case

$$f(t) = \begin{cases} f_0 \cos \omega t & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

c) Compute the expectation value of the total energy gained by the oscillator at resonance, $\omega = 1$, if $|\psi(0)\rangle = |0\rangle$, its unperturbed ground state. Sketch your result as a function of T.

(You may wish to work with creation/annihilation operators to make the algebra more familiar.)