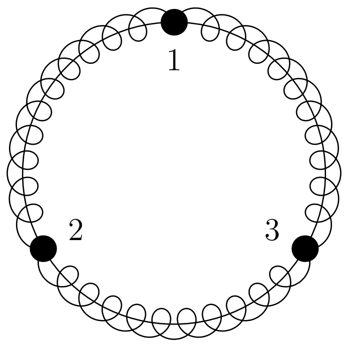
M02Q.2—Three Particles on a Ring

Problem

Consider a molecule of three particles, each carrying $S = \frac{1}{2}$ connected by springs and constrained to move on a circle of circumference L. (The last bit is to make the geometry easier, the situation without this constraint works out the same way.)



The Hamiltonian for this system is

$$H = 2J \sum_{i=1}^{3} (1 - \alpha x_{i,i+1}) \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{2} k x_{i,i+1}^2$$

where the $\boldsymbol{S}_i$ are spin-1/2 operators and the $x_{i,i+1}\,(> 0)$ are the distances along the circle between neighboring particles i and $i+1$. By periodicity, $4 \equiv 1$. Finally, $J > 0$.

a) Find the ground states of the spin system in isolation, i.e. when $\alpha = 0$. How many ground states are there?

b) Find the ground states of the coupled system when $0 < \alpha \ll 1$. Restrict yourself to ground states in which two of the three interparticle distances are identical, e.g. $x_{1,2} = x_{2,3}$.

(You may find the following identity useful in the computations in part b):

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = \frac{1}{4}(2P_{ij} - 1)$$

where P_{ij} interchanges spins i and j.)