M03M.2—Rolling Sphere

Problem

The problem of a sphere rolling without slipping directly down an inclined plane is an old chestnut of freshman physics. In this problem, you are asked to examine more general motions of a sphere rolling without slipping on inclined and rotating planes.

The state of motion of a sphere of mass M is specified by giving its position $\vec{R}$ and velocity $d\vec{R}/dt$ with respect to a fixed point and its angular velocity $\vec{\omega}$ with respect to its center of mass. The rolling without slipping constraint is usually expressed as $d\vec{R}/dt = a\vec{\omega} \times \hat{n}$ where a is the sphere's radius and $\hat{n}$ is a unit vector normal to the plane on which the sphere is rolling. There is a constant force $\vec{f}$ which acts in the plane to enforce the constraint and this constraint force has to be taken into account in the equations for acceleration of the center of mass and for the time rate of change of the angular momentum $I\vec{\omega}$ about the center of mass.

a) Suppose that the plane is inclined to the horizontal at an angle θ in the Earth's gravitational field. Show that you can eliminate the constraint force to find an equation for the center of mass alone and show that the center of mass experiences an acceleration of $\frac{5}{7}g\sin\theta$ along the 'downhill' direction of the plane. Hence, the trajectories of this rolling sphere are parabolae.

b) Now consider the case that the pane is not inclined, but is rotating with angular velocity $\vec{\Omega}$ about the vertical axis. The most important modification to the calculation you have just done is that the condition for rolling without slipping changes. Use the changed condition in your previous analysis to show that the center of mass executes circular motion in the horizontal *inertial* frame with frequency $\frac{2}{7}\Omega$! In other words, the rolling sphere executes Larmor orbits, just like a charged particle in a uniform magnetic field!

For reference, note that the moment of inertia of a homogeneous sphere is $\frac{2}{5}Ma^2$.