M03E.3—Electromagnetic Wave in a Plasma

Problem

The $\boldsymbol{E}$-vector of a plane electromagnetic wave propagating along the z-axis and having a polarization vector $\vec{e} = (\alpha_x, \alpha_y, 0)$ can be written

$$\vec{E}(\vec{r}, t) = E(\alpha_x \hat{x} + \alpha_y \hat{y}) e^{i(kz - \omega t)}$$

(the polarization vector is taken to be of unit length, $\vec{\alpha} \cdot \vec{\alpha}^* = 1$). In free space, the dispersion relation is $\omega = ck$ and the wave propagates with both phase and group velocity equal to c.

Now let the wave propagate through a dilute plasma containing a density N of free mobile electrons of mass m and charge e (along with a background of compensating positive charge taken to be so massive as to be fixed in place). By solving for the motion of an electron in the electric field of the propagating wave (i.e. ignoring the effect of the wave's B-field on the motion) one can infer a polarization density $\vec{P} \propto \vec{E}$. This in turn allows us to infer a frequency-dependent dielectric constant via

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon(\omega) \vec{E}.$$

The index of refraction of plasma is then given by $n(\omega) = \sqrt{\epsilon/\epsilon_0}$.

a) Use this line of argument to compute the frequency dependence of the index of refraction $n(\omega)$ of a plasma. Turn your result into a dispersion relation $\omega(k)$ and find the limiting frequency ω_p as the wavelength of the wave goes to infinity. This cutoff frequency, below which waves cannot propagate, is called the plasma frequency.

b) Now let the plasma be subject to a static magnetic field $B_0 \hat{z}$ in the direction of propagation of the wave. Also assume that the electrons have some kinetic energy so that, in the absence of any other perturbation, they execute circular motion about the static magnetic field lines at the Larmor frequency $\omega_L = |eB_0|/m$. Extend your calculation of the dielectric constant of the plasma to this new case. The response is different for different states of circular polarization so, for definiteness, analyze the case of right circular polarization of the propagating wave ($\vec{\alpha} = (\hat{x} + i\hat{y})/\sqrt{2}$).

c) Find the lowest frequency at which such a wave can propagate. Express your answer in terms of ω_p and ω_L.