M03Q.3—Addition of Angular Momentum

Problem

The total angular momentum of a valence electron in an atom of orbital angular momentum l can be either $J = l + 1/2$ or $J = l - 1/2$. The two total angular momentum states are typically split by spin-orbit interactions, leaving $2J + 1$ degenerate states of the same J, but different J_z. Upon applying a uniform magnetic field $B\hat{z}$, these magnetic substates are split by the interaction between the magnetic moment and the applied magnetic field. The perturbation Hamiltonian is

$$H_{int} = -\frac{e\hbar}{2m_e c}B(L_z + 2S_z) = -\frac{e\hbar}{2m_e c}B(J_z + S_z)$$

where the relative factor of two between L_z and S_z is due to the famous fact that the g-factor of the electron is two. The Zeeman splitting of the degenerate multiplet of total angular momentum J is thus

$$\delta E_{J,J_z} = -\frac{e\hbar B}{2m_e c}\langle J, J_z|(J_z + S_z)|J, J_z\rangle = -\frac{e\hbar B}{2m_e c}(J_z + \langle J, J_z|S_z|J, J_z\rangle).$$

To evaluate this explicitly we need to solve the slightly nontrivial problem of evaluating the matrix elements of S_z.

a) Consider the special case $l = 1$. Construct the $J = 3/2, 1/2$ states by angular momentum addition and evaluate $\langle J, J_z|S_z|J, J_z\rangle$ for all the states. Show that the energy splittings satisfy $\delta E_{J,J_z} = g_J J_z$ and state the two values of g_J that you have just computed.

b) Now consider the case of general l. It is possible, but tedious, to directly verify that $\delta E = g_J^l J_z$ for the two possible J multiplets. Assuming that this is so (i.e. that it suffices to compute g_J^l in any magnetic sublevel J_z), find the relevant g_J^l factors for the two multiplets $J = l \pm 1/2$.

c) The above calculations are particular examples of a general result most easily proved using the Wigner-Eckart theorem. For a general vector operator $\vec{A}$ (one whose commutation relations with $\vec{J}$ are of the form $[J_i, A_j] = i\epsilon_{ijk}A_k$) the theorem says that

$$\langle J, M'|\vec{A}|J, M\rangle = \frac{\langle J|\vec{J}\cdot\vec{A}|J\rangle}{J(J+1)}\langle J, M'|\vec{J}|J, M\rangle.$$

Use this theorem to rederive the g_J^l factor for $J = l + 1/2$ multiplet which you computed above.