M03T.3—Particles in a Box

Problem

Consider a single free particle of mass m confined to a volume V. Let $Z_1(m)$ denote the quantum partition function for this system (where the partition sum is taken over the discrete energy levels of a particle of mass m in a box of volume V).

a) Show that $Z_1(m) \to V/\lambda^3$ with $\lambda = h/\sqrt{2\pi mkT}$ in the classical (or small $\hbar$) limit. Use this result to calculate the classical energy and heat capacity at fixed volume of the single particle system.

b) Identify the temperature at which this approximation breaks down.

c) Now consider a system consisting of two identical, non-interacting particles in the same box. Because of the effects of identical particle statistics, the classical expectation for the two-particle partition function $Z_2(m) = Z_1(m)^2$ is not quite correct. Show that the exact free boson and free fermion two-particle partition sums can in fact be expressed in a simple way in terms of the one-particle functions $Z_1(m)$ and $Z_1(m/2)$.

d) Using the classical approximation $Z_1(m) = V/\lambda^3$ derived in the first part of this problem, calculate the correction to the energy E and the heat capacity C due to Bose or Fermi statistics.