M04Q.2—Positronium in a Magnetic Field

Problem

Consider ths spin degrees of freedom of a two-particle system, one with spin S and the other with spin 1/2. The Hamiltonian is

$$H = a\boldsymbol{S_1} \cdot \boldsymbol{S_2},$$

with a a constant. Here $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ stand for the vector spin operators of particle 1 and 2, respectively.

a) Calculate the eigenvalues of H. What are their multiplicities?

b) Consider now the special case, corresponding to the spin degrees of freedom of positronium, where both spins are $S = 1/2$. What are the eigenvalues and corresponding eigenstates of H?

c) When the positronium is placed in a magnetic field, oriented in the z-direction, the Hamiltonian becomes

$$H_B = a\boldsymbol{S_1} \cdot \boldsymbol{S_2} + b(S_1^z - S_2^z),$$

where b is a constant. Describe the multiplicities of the resulting spectrum, and calculate the eigenvalues and eigenvectors of H_B.