M05Q.1 - Hydrogen Atom in an Electric Field

Problem

a) A spatially uniform, time-independent electric field $\boldsymbol{E}$ is applied to a hydrogen atom for which the electron was initially in its ground state $\phi_g = e^{-r}/\sqrt{\pi}$. The field is much too small to ionize the atom in any reasonable time, but it slightly distorts the electron charge distribution. The distorted electron wave function is $\psi = \psi(\boldsymbol{r})$, where $\boldsymbol{r}$ is the displacement (in units of the Bohr radius a_B) of the electron from the nucleus. You can ignore the spins of the electron and the nucleus. Since the polarized atom experiences no net force in the uniform field $\boldsymbol{E}$, the quantum-mechanical expectation value $\boldsymbol{E}' = (e/a_B^2)\langle\psi|e\boldsymbol{r}/r^3|\psi\rangle$, of the electric field produced at the nucleus by the distorted electron charge distribution must cancel the applied field. Prove that this is so.

b) The applied electric field oscillates along the z axis with amplitude $E\cos\omega_{eg}t$, where $\omega_{eg} = (E_e - E_g)/\hbar$ is the Bohr frequency of the transition from the $1s$ ground state, of energy E_g, to the $2p$ excited state, of energy E_e. Find $\boldsymbol{E}'$ in this case. Assume that $eEa_B \ll \hbar/\tau$, where the natural radiative lifetime of the $2p$ state is given by

$$\frac{1}{\tau} = \frac{8\pi e^2 \omega_{eg}^3 a_B^2 z_{eg}^2}{3hc^3}$$

The matrix element of electron displacement along the z axis is,

$$z_{eg} = \int \phi_e^* z \phi_g d^3 r,$$

and e is the elementary charge.

HINT: One of these problems has a solution that does not rely on perturbation theory.
Possibly useful hydrogen wave function (in atomic units) for the first excited p state of axial symmetry:

$$\psi_e = \frac{ze^{-r/2}}{\sqrt{32\pi}}$$

Possibly useful definite integral for $a > 0$:

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}.$$