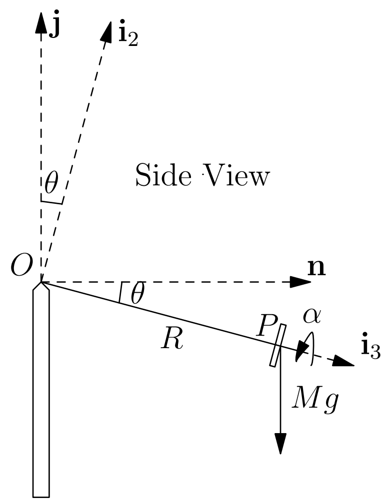
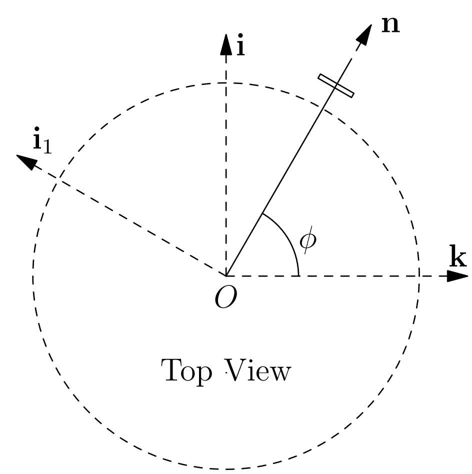
J06M.1 - Gyroscope

Problem

A gyroscope, illustrated in the figures below, is free to pivot about point O under the effect of gravity. Its total mass is M and its center of mass is located at point P at a distance R from O. In the reference frame $(O; \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ of the gyroscope (see figures), its moment of inertia tensor about point O is $\hat{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$. If $(O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is the laboratory frame and $\boldsymbol{n}$ that axis at the intersection between the plane $\boldsymbol{i}_2\boldsymbol{i}_3$ and the plane $\boldsymbol{ik}$, define α to be the rotation angle of the gyroscope around $\boldsymbol{i}_3$, θ (the nutation angle) to be the angle between $\boldsymbol{i}_3$ and $\boldsymbol{n}$ and ϕ (the precession angle) as the angle between $\boldsymbol{k}$ and $\boldsymbol{n}$.





a) Write the Lagrangian of the system and its energy in terms of the angles α, θ, ϕ and of their time derivatives. [*Hint:* In order to find the expression for the kinetic energy $\frac{1}{2}\vec{\omega} \cdot \hat{I}\vec{\omega}$, first write $\vec{\omega}$ in the reference frame $(O; \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$.]

b) Write the conservation laws for this system: energy and two projections of angular momentum.

c) From the conservation laws deduce a closed equation for θ in the form $F(\dot{\theta}, \theta) = 0$.

d) At time $t = 0$ the gyroscope is place horizontally ($\theta = 0$) with zero nutation angular velocity ($\dot{\phi} = \dot{\theta} = 0$) and spin angular velocity $\dot{\alpha} = L_0/I_3$. Show that for $\theta \ll 1$ the previous equation and these initial conditions admit an approximate solution $\theta = \theta_0(1 - \cos\omega_n t)$. Compute the frequency ω_n, the amplitude θ_0, and the average precession velocity $\langle\dot{\phi}\rangle$. Find the condition on the initial data (i.e. on L_0) for which $\theta \ll 1$ remains a good approximation at all times.