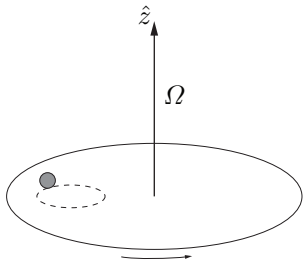
J07M.1 - Ball on a Turntable

Problem



A spherically symmetric ball of mass m, moment of inertia I about any axis through its center, and radius a, rolls without slipping and without dissipation on a horizontal turntable (so frictional forces act on the ball at its point of contact with the turntable, but do no work). The turntable is rotating about the vertical z-axis at constant angular velocity $\Omega\hat{\boldsymbol{z}}$.

Use a laboratory frame coordinate system $(\boldsymbol{r}, z), \boldsymbol{r} = (x, y)$ so the center of the ball is in the plane $z = 0$, and the axis of the turntable is $\boldsymbol{r} = 0$. Distinguish the (vector) angular velocity $\boldsymbol{\omega}_{rot}$ of the rotation of the ball about its center, and the angular velocity $\omega_{cm}\hat{\boldsymbol{z}}$ of the orbit about its center, where ω_{cm} is scalar.

a) Find a solution of the equations of motion where the center of the ball is stationary at a point $\boldsymbol{r}_0 \neq 0$ in the laboratory frame. What is $\boldsymbol{\omega}_{rot}(t)$ for this solution?

b) Find the general solution for the orbit $\boldsymbol{r}(t)$ of the center of the ball, when the initial center-of-mass position and velocity are $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$. Show that, in the laboratory frame, the ball rolls along a circular path (in general centered at a point $\boldsymbol{r} \neq 0$), with an angular velocity ω_{cm} that is independent of the initial conditions.

c) If the ball is solid with uniform density, what is the relation of ω_{cm} to Ω?