M07Q.1 - Quantum Virial Theorem

Problem

Let $F(\boldsymbol{r}, \boldsymbol{p})$ be some function of position and momentum without explicit time dependence, $\partial F/\partial t = 0$.

a) If $|\psi_n\rangle$ is an eigenstate of a Hamiltonian H in the Schrödinger representation, show that

$$\frac{d}{dt}(\langle\psi_n|F|\psi_n\rangle) = 0.$$

b) Suppose

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}).$$

Show that

$$\langle\psi_n|\frac{\boldsymbol{p}^2}{2m}|\psi_n\rangle = \frac{1}{2}\langle\psi_n|\boldsymbol{r} \cdot \nabla V(\boldsymbol{r})|\psi_n\rangle.$$

c) Use this quantum-mechanical version of the virial theorem to estimate the fraction of the proton rest mass that is in the form of potential energy. The gluon-mediated force between two quarks is nearly independent of the distance between them. The rest mass of the quarks is much smaller than the mass of the proton, so strictly speaking one should use a relativistic version of the virial theorem. However, the non-relativistic version still gives approximately correct results, see “Relativistic virial theorem”, Phys. Rev. Lett. **64**, 2733-2735 (1990).