M07M.2 - Particle in an Ellipsoid

Problem

Consider a point mass m constrained to move without friction on the surface of an ellipsoid. There is no gravity in this problem. The coordinates of the mass can be parametrized by the following equations:

$$x = a\sin\theta\cos\phi \quad y = a\sin\theta\sin\phi \quad z = b\cos\theta.$$

a) Write the Lagrangian using $(\theta, \phi, \dot{\theta}, \dot{\phi})$ coordinates and derive the equations of motion.

b) Show that one period of the motion is given by

$$T(E, A) = 2\int_{\theta_-}^{\theta_+} \frac{d\theta}{\sqrt{-V_{E,A}(\theta)}}$$

where

$$V_{E,A}(\theta) = \frac{ma^2A^2 - 2E\sin^2\theta}{m\sin^2\theta(a^2\cos^2\theta + b^2\cos^2\theta)}$$

and $\theta_\pm$ are the two roots of the equation $V_{E,A}(\theta) = 0$. Here E is the energy and A is the conserved quantity $A = \dot{\phi}\sin^2\theta$.