M07T.3 - 3D Ising Model

Problem

The Ising model on a 3-dimensional square lattice with spin-1/2 particles is defined by the Hamiltonian

$$H = -J \sum_{\boldsymbol{i},\boldsymbol{j}} \sigma_{\boldsymbol{i}} \sigma_{\boldsymbol{i}+\boldsymbol{j}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}}$$

where $J > 0$, $\boldsymbol{i}$ labels sites of the 3-dimensional lattice, $\boldsymbol{j}$ runs over nearest neighbor sites in 3 dimensions and $\sigma_{\boldsymbol{i}}$ is equal to $+1$ or -1.

The Ising model is often solved using the *mean field approximation*, consisting of replacing the spin interaction Hamiltonian by the mean field interaction

$$H_m = -M \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}}$$

where M is a parameter fixed by the self-consistency condition to be $M = 6J\langle\sigma_{\boldsymbol{i}}\rangle$.

a) For the Hamiltonian H_m calculate the free energy, entropy, and $\langle\sigma_{\boldsymbol{i}}\rangle$ at temperature T.

b) Show that for $B = 0$ at low temperature a self-consistent solution with $\langle\sigma_{\boldsymbol{i}}\rangle \neq 0$ has a lower free energy than a solution with $\langle\sigma_{\boldsymbol{i}}\rangle = 0$.

c) Find the critical temperature T_c above which the spontaneous magnetization vanishes at zero external field B.

d) How can one build a refrigerator using the spins as the working substance? Describe qualitatively how one can efficiently cool a substance by manipulating the spin degrees of freedom which obey this mean field theory.