J08M.3 - Fluid Flow

Problem

When we derive Newton's equations of motion from a Lagrangian or Hamiltonian, the equations are invariant under time reversal, so that if $x(t)$ is a solution, so is $x(-t)$. If we add terms corresponding to damping or viscosity, the invariance is broken, and motions become obviously irreversible. Strangely, a form of reversibility is restored for fluid motion in the limit that viscosities are very large.
Consider a fluid with viscosity η and density ρ, and assume that it is incompressible. The equations of motion are the Navier-Stokes equations,

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right] = \vec{\nabla} p + \eta \nabla^2 \vec{v}$$
$$\vec{\nabla} \cdot \vec{v} = 0\,,$$

where $\vec{v}(\vec{x}, t)$ is the velocity of the fluid element at position $\vec{x}$ at time t, and $p(\vec{x}, t)$ is the pressure. To be concrete, imagine that we have a layer of fluid between two (large) parallel plates, a distance d apart.

a) Let one of the plates move at velocity v_0, with the other plate held fixed. Now the natural unit of length is d, the natural unit of velocity is v_0, and the natural unit of pressure is ρv_0^2. Show that, in these natural units, a single term in the Navier-Stokes equations becomes dominant at large viscosity. Since viscosity has units, "large" means large relative to some characteristic scale η_c, which you should determine.

b) In this limit of large viscosity (usually called the "low Reynolds number" limit, $\mathrm{Re} \equiv \eta_c/\eta$), show that if the plate moves for a time T with velocity v_0, and then with velocity $-v_0$ for an equal time T, all elements of the fluid will be returned exactly to their initial locations, so that motion is reversible. You should show this explicitly for the problem of fluid between two plates (by solving the equations), and give a more general argument (which doesn't require solving the equations).

Notes: Recall that fluid immediately adjacent to a wall must move with the velocity of the wall - no slipping. The reversibility of motion means that if we inject a blob of dye into the fluid, then the motion of the wall at v_0 will spread the dye out (you should think about why) but then the motion at $-v_0$ will reassemble the original blob.