M08Q.2 - Ladder Operators

Problem

The dynamics of a system is characterized by the Hamiltonian

$$H = a^\dagger a + \frac{1}{2}, \quad [a, a^\dagger] = 1\,.$$

a)   Show that the ground state of this system, $|0\rangle$, satisfies

$$a|0\rangle = 0\,.$$

b)   Consider the state

$$|\alpha\rangle = \mathcal{N}e^{\alpha a^\dagger - \alpha^* a}|0\rangle\,,$$

      where $\mathcal{N}$ is some normalization constant. Show that $a|\alpha\rangle = \alpha|\alpha\rangle$. Find $\mathcal{N}$.

c)   Consider the change of variables

$$a = \frac{1}{\sqrt{2}}(q + ip), \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip)\,.$$

      Derive and interpret the hamiltonian in this set of new variables.

d)   Calculate $\langle\alpha|q|\alpha\rangle$. Describe the time dependence of $\langle\alpha|q|\alpha\rangle$.