J10Q.3 - Spin-Dependent Scattering

Problem

Consider the usual Hamiltonian for non-relativistic electrons moving in 2D:

$$H = H_0 + V(x,y)\,, \quad \text{where} \quad H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}\,.$$

Electrons experience a "step" potential $V(x,y) = 0$ for $x < 0, V = V_0 > 0$ fro $x > 0$.

a) Electrons arriving from the region $x < 0$ are incident normally in the step (*i.e.*, have conserved momentum $p_y = 0$). Find the probability of reflection.

Now consider a similar problem, but this time one where the Hamiltonian couples the spatial and spin degrees of freedom of the electron in an essential way:

$$\boldsymbol{H} = \boldsymbol{H}_0 + V(x,y)\boldsymbol{\sigma}^0\,, \quad \text{where} \quad \boldsymbol{H}_0 = v_F(\boldsymbol{\sigma}^x p_y - \boldsymbol{\sigma}^y p_x)\,,$$

where $\boldsymbol{\sigma}^i$ are 2×2 Pauli matrices and $\boldsymbol{\sigma}^0$ is the identity matrix; v_F is a characteristic speed, and $V(x,y)$ is the same "step" potential as in a), above.
In this problem, eigenstates of $\boldsymbol{H}_0$ with momentum $\boldsymbol{p}$ have energie $\varepsilon_\pm(\boldsymbol{p}) = \pm v_F|p|$. They are non-degenerate for $\boldsymbol{p} \neq 0$.

b) Find the two-component wavefunction $\Psi_\sigma(x,y)$ of *positive energy* eigenstates of $\boldsymbol{H}_0$ with momentum $\boldsymbol{p} \equiv (p_x, p_y) = |p|(\cos\theta, \sin\theta)$. (The index σ labels the two possible values of the z-component of spin.)

c) Electrons arriving from the region $x < 0$ with $p_y = 0$ (as in part a)) now have 100% probability of transmission through the step. Explain why.

d) Consider electrons arriving with energy $E = 2V_0$ and $p_y = |p|\sin\theta$:

   i. For what range of θ is transmission through the step possible? (Hint: an analog of "Snell's law" relates angles of incidence and refraction, θ and θ'.)

   ii. In this range, find the reflection probability $R(\theta)$.

Note 1: The second Hamiltonian only requires that both wavefunction components $\Psi_\sigma(x,y), \sigma =\uparrow,\downarrow$, are continuous at the step, with no condition on their derivatives. (It describes electrons on the surface of a "topological insulator".)

Note 2: The Pauli matrices are:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \boldsymbol{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$