Section A. Quantum Mechanics

1. Energy of a large atom

This problem is about large neutral atoms with large nuclear charge Z. Treat the nucleus as an stationary, infinitely-heavy point charge, located at the origin in space. Take the atom to be neutral, so the number of electrons is $N = Z$. As an approximation, ignore relativistic effects (which limit Z to less than 137); also **neglect the electron-electron Coulomb repulsion** and consider only the attraction of the electrons to the nucleus.

(a) Explicitly, what is the Hamiltonian H for this problem, and what are the energies of its electronic orbitals?

(b) Give expressions for the ground state energy E_n, and for the number of electrons $Z_n = N = Z$, when exactly n "closed shells" of electronic orbitals are filled. From the leading asymptotic behavior of Z_n and E_n for large n, obtain the leading asymptotic behavior of $E(Z)$ for large Z.

(c) $E = K + V$, where K is the kinetic energy and V the potential energy. In terms of E, what are the expectation values of K and V?

(d) If r_i, $i = 1, \ldots, N$, are the radial coordinates of the electrons, an "average distance from the nucleus", may be defined by the expectation value

$$\frac{1}{r_{\text{av}}} = \langle \frac{1}{N} \sum_{i=1}^{N} \frac{1}{r_i} \rangle.$$

How does r_{av} depend on Z in the large-Z limit?