2. ANHARMONIC OSCILLATOR

A non-relativistic particle with mass m moves one-dimensionally in the potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 + \lambda x^4, \quad \text{with } \lambda > 0.$$

Let $|\Psi_0(\lambda)\rangle$ be the ground state of this system, and $E_0(\lambda)$ be the ground-state energy. For small λ, the quartic term in the potential can be treated as a small perturbation of the $\lambda = 0$ harmonic oscillator problem, which has an oscillation frequency ω.

(a) The particle coordinate x can be expressed as an operator in terms of $a^\dagger$ and a, the raising and lowering operators for the $\lambda = 0$ harmonic oscillator problem, where $a|\Psi_0(\lambda = 0)\rangle = 0$. Give such an expression for x.

(b) Compute the perturbation expansion of the ground-state energy $E_0(\lambda)$ up to first order in λ.

(c) Again up to first order in λ, compute the perturbation expansion of the ground-state expectation value $\langle\Psi_0(\lambda)|x^2|\Psi_0(\lambda)\rangle$.

In the opposite limit of *large* positive $\lambda \to \infty$, the leading behavior of the ground state energy $E_0(\lambda)$ will be proportional to λ^α where α is a positive exponent.

(d) (Up to an undetermined numerical multiplicative factor) find the asymptotic large-λ behavior of the ground-state energy $E_0(\lambda)$, giving the explicit value of α.

(You may find a simple variational estimate of $E_0(\lambda)$, or scaling arguments, helpful in part (d).)