3. **Precession of Equinoxes**. The Sun and Moon exert torques on the Earth that cause its rotation axis to precess around the normal to the plane of the ecliptic once every 26000 years. The component of the gravitional potential from the Sun that gives rise to a torque on the Earth is:

$$V = \frac{GM\left(I_3 - I_1\right)}{2r^3}\left[\frac{1}{2} - \frac{3}{2}\cos^2\theta\right], \tag{2}$$

where M is the mass of the Sun, r is the radius of the Earth's orbit, I_1 and I_3 are moments of inertia around Earth's principle axes, and θ is the tilt of the Earth's rotation axis relative to the normal to the plane of the ecliptic. For orientation, the angular frequency of rotation of the Earth is ω_3, and the potential has been averaged over one orbital period since the precession period is much longer than a year.

a) Write down the Lagrangian describing the Earth's rotational and orbital motion in the graviational field of the Sun assuming no nutation of the rotation axis ($\dot{\theta} = \ddot{\theta} = 0$).

b) Assuming the precession frequency $\dot{\phi}$ is much less than the rotational frequency ω_3 of the Earth ($\dot{\phi} << \omega_3$), derive a formula for the ratio $\dot{\phi}/\omega_0$, where ω_0 is the angular frequency of Earth's orbital motion. For full credit your answer should be in terms of ω_0, ω_3, I_1, I_3, and θ only.

c) Looking down on the northern hemisphere from above the plane of the ecliptic, in which direction does the Earth's rotation axis precess?

d) Using $\left(I_3 - I_1\right)/I_3 = 0.003$, $\theta = 23.5°$, and the known frequencies of Earth's rotation and orbit, give a rough estimate for the precession period in years. [Note that you should obtain a value larger than 26000 years since the torque from the Moon is larger than, and adds to, that from the Sun.]