Problem 0, Page

Section B. Statistical Mechanics and Thermodynamics

1. Consider a liquid placed in a very wide container that is in thermal equilibrium at temperature T with its surroundings. Let $z(\vec{r})$ be the height of the liquid at point $\vec{r} = (x, y)$ defined such that the equilibrium height in absence of thermal fluctuations is $z(\vec{r}) = 0$. For small deviations around the equilibrium, the potential energy is approximately

$$E_{\rm pot} \approx E_0 + \frac{1}{2}\int dxdy\left[\sigma\left(\frac{\partial z}{\partial x}\right)^2 + \sigma\left(\frac{\partial z}{\partial y}\right)^2 + \rho g z^2\right],$$

where E_0 is a constant, σ is the surface tension, ρ is the difference between the density of the liquid and that of the gas, and g is the gravitational acceleration.

   (a) For a periodic box of side length L, express the potential energy $E_{\rm pot}$ in terms of the Fourier coefficients $A(\vec{k})$ defined by

$$z(\vec{r}) = \frac{1}{L}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} A(\vec{k}),$$

   where $A(-\vec{k}) = A(\vec{k})^*$ and $\vec{k} = (k_x, k_y) = \frac{2\pi}{L}(n_x, n_y)$ (with n_x and n_y integers).

   (b) Due to thermal fluctuations,

$$\left\langle |A(\vec{k})|^2 \right\rangle = \frac{1}{a\vec{k}^2 + b},$$

   as long as $|\vec{k}|$ is below a certain cutoff. What are the values of a and b at temperature T, in terms of the model's parameters (σ, ρ, T, L)?

   (c) Find an approximate expression for the r.m.s. width $W = \sqrt{\langle z(\vec{r})^2\rangle}$, for wide containers, in terms of a, b, and the maximal value $k_{\rm max}$ of $|\vec{k}|$. Assume also that $k_{\rm max}^2 \gg b/a$.

   Hint: modes with different wavevectors are not correlated, and thus $\left\langle A(\vec{k})A(\vec{k}')^* \right\rangle = 0$ if $\vec{k} \neq \vec{k}\,'$.

   (d) What determines $k_{\rm max}$?