3. Consider a quantum system consisting of a harmonic oscillator that is coupled to a spin-1/2 particle. The Hamiltonian is given by

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\Omega S_z + \hbar g(aS_+ + a^\dagger S_-) \tag{1}$$

where $a, a^\dagger, S_z, S_+, S_-$ are the usual quantum operators for a harmonic oscillator and spin-1/2 particle.

When $g = 0$, the eigenstates of the Hamiltonian can be labeled by $|n, \pm\rangle$, where n is the harmonic oscillator occupation number and $+$ and $-$ refers to spin up and down states.

(a) Determine which of the uncoupled states of the Hamiltonian mix together when $g \neq 0$.

(b) Find the eigenstates of the Hamiltonian when $g \neq 0$ without making any assumptions about the relative size of the various terms in (1).

(c) Make a sketch of how the energy levels change as a function of Ω in the range $0 < \Omega/\omega < 2$. Assume moderate coupling strength $g < \omega$.