2. A hydrogen atom located at $\vec{r} = 0$ is initially in the $2s$ state when an ion of charge Q passes by it. Assume the ion moves with constant velocity $\vec{v} = v\hat{y}$ on a straight line whose closest approach to the hydrogen atom is $\vec{b} = b\hat{z}$, with $b \gg a_B$, where a_B is the Bohr radius. While the ion passes by, the electron in the atom experiences a time-dependent potential

$$V_1(\vec{r}, t) = \frac{Qe}{|\vec{b} + \vec{v}t - \vec{r}|}\,.$$

We are interested in calculating the transition probability to one of the $2p$ states. In this problem, you may assume that the $2s$ and $2p$ states are degenerate.

(a) Find an expansion of $V_1(\vec{r}, t)$ that is valid at all times and that is appropriate for calculating the transition probability in the limit $b \gg a_B$. Identify the leading term in this expansion that will give a non-vanishing transition amplitude between the $2s$ and at least one of the $2p$ states in first-order time-dependent perturbation theory.

(b) Using first-order time-dependent perturbation theory, calculate to leading order in a_B/b the probability that the atom winds up in a $2p$ state.

Some useful hydrogen atom wave-functions are:

$$\phi_{2s} = \frac{1}{2\sqrt{2\pi a_B^3}}\left(1 - \frac{r}{2a_B}\right) e^{-r/(2a_B)}\,,$$

$$\phi_{2p,0} = \frac{z}{4\sqrt{2\pi a_B^5}} e^{-r/(2a_B)}\,,$$

$$\phi_{2p,\pm1} = \frac{x \pm iy}{8\sqrt{\pi a_B^5}} e^{-r/(2a_B)}\,.$$

You may also find the following integral useful:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{3/2}} = 2\,.$$